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Periodic coefficients

$$X' = A(t)X$$

$$A(t+\omega) = A(t)$$

note that $X'(t+\omega) = A(t+\omega)X(t+\omega)$

$$= A(t)X(t+\omega)$$

$\therefore Y(t) = X(t+\omega)$ is also a solution.

lemma:

if $C \in M_n(\mathbb{C})$ is invertible, $\exists B \in M_n(\mathbb{C})$, such that $e^B = C$

we know that $J = Q^{-1}CQ$, Jordan canonical form

$J = \text{diag}(J_1, \dots, J_r)$

$\therefore e^K = \text{diag}(e^{K_1}, \dots, e^{K_r}) = \text{diag}(J_1, \dots, J_r) = J$

$$J_i = (a) \quad \text{or} \quad J_i = \begin{pmatrix} \mu & 1 & & 0 \\ & \mu & 1 & \\ & & \mu & \ddots \\ 0 & & & \mu \end{pmatrix}$$

$a \neq 0$, $\mu \neq 0$ because C is invertible

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if $J_i = (a)$, then $K_i = \log(a) = \log(|a|) + i\theta$

if $J_i = \begin{pmatrix} \mu & & 0 \\ & \ddots & \\ 0 & & \mu \end{pmatrix}$,

$J = \mu(I_s + N/\mu)$

$\quad$ nilpotent, $N^s = 0$

note that $\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} = \sum (-1)^{k+1} \frac{z^k}{k}$

$$e^{\log(1+x)} = 1 + x = 1 + \left(z - \frac{z^2}{2} + \frac{z^3}{3} + \dots\right) + \left(z - \frac{z^2}{2} + \frac{z^3}{3} + \dots\right)^2 + \dots$$

similarly, $\log(I_s + \frac{N}{\mu}) = \frac{N}{\mu} - \frac{1}{2}\left(\frac{N}{\mu}\right)^2 + \frac{1}{3}\left(\frac{N}{\mu}\right)^3 + \dots$

$\quad$ finite sum, $\because N^s = 0$

$$e^{\log(I_s + \frac{N}{\mu})} = I_s + \left(\frac{N}{\mu} - \frac{1}{2}\left(\frac{N}{\mu}\right)^2 + \dots\right) + \left(\frac{N}{\mu} - \frac{1}{2}\left(\frac{N}{\mu}\right)^2 + \dots\right)^2 + \dots + \left(\frac{N}{\mu} - \frac{1}{2}\left(\frac{N}{\mu}\right)^2 + \dots\right)^{s-1}$$

$$= I_s + \frac{N}{\mu}$$

$$\therefore e^{\log(\mu)} e^{\log(I_s + \frac{N}{\mu})} = e^{l(\frac{N}{\mu})} = \mu\left(I_s + \frac{N}{\mu}\right) = J$$

$\therefore e^K = J$

$\therefore C = QJQ^{-1} = Qe^K Q^{-1} = e^B \quad B = QKQ^{-1}$ //

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since $X(t+\omega)$ and $X(t)$ are solutions,

$$X(t+\omega) = X(t)C = X(t)e^{R\omega}$$

let $P(t) = X(t)e^{-Rt}$, note that $P(t+\omega) = X(t+\omega)e^{-R(t+\omega)}$
$= X(t)e^{-Rt} = P(t)$

$\because X' = AX$

$$P'e^{Rt} + PRe^{Rt} = APe^{Rt}$$

$$\therefore P' + PR = AP$$

let $X = PY$, $P'Y + PY' = APY = P'Y + PRY$

similar to exponential

$$\therefore Y' = RY$$

$$P(t+\omega) = P(t) = X(\omega+t)e^{-R(t+\omega)}$$
$$= Y(t)e^{-R(t+\omega)}$$
$$= X(t)e^{-Rt}$$

Multipliers

let $\tilde{X}, \tilde{X}_1$ be two bases of solution $X' = AX$

$$\tilde{X}_1 = \tilde{X}T$$

$$\tilde{X}_1(t+\omega) = \tilde{X}_1(t)C_1$$

$$\therefore C_1 = \tilde{X}_1^{-1}(0)\tilde{X}_1(\omega) = T^{-1}\tilde{X}^{-1}(0)\tilde{X}(\omega)T = T^{-1}(0)\tilde{X}^{-1}(0)\tilde{X}(0)CT$$
$$= T^{-1}CT$$

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$\therefore C_1 \sim C$ same eigenvalues $\mu_1, \cdots, \mu_n$

note that iff $X(t+\omega) = \mu X(t)$,

~~then~~ $X(t+\omega) = \tilde{X}(t+\omega)\xi = \tilde{X}(t)C\xi = \mu X(t)$
$= \mu \tilde{X}(t)\xi$

$\therefore C\xi = \mu\xi \iff X(t+\omega) = \mu X(t)$

if $\tilde{X}(0) = I_n$, then $\tilde{X}(\omega) = \tilde{X}(0)e^{R\omega} = e^{R\omega} = C$

$\therefore \tilde{X}(\omega)\xi = \mu\xi$

moreover, let $\{\xi(\mu) \mid X(t+\omega) = \mu X(t)\}$

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Analytic Coefficients

$$X' = AX$$
$$A = A(t)$$

Normed vector spaces:

$$S = \sum_{i=0}^{\infty} c_i \qquad S_m = \sum_{i=0}^{m-1} c_i$$

~~$\lim S_m$~~

$$\lim_{m\to\infty} S_m = S$$

define norm $\| \ \|$ over V, $X \in V$:

$$\|X\| \geq 0$$
$$\|X\| = 0 \text{ iff } X = 0$$
$$\|X + Y\| \leq \|X\| + \|Y\|$$
$$\|\alpha X\| = |\alpha| \|X\| \quad \alpha \in \text{ some field}$$

let $Y := -Y$,

$$\|X + (-Y)\| = \|X - Y\| \leq \|X\| + \|Y\|$$

let $Y := -X + Y$,

$$\|Y\| - \|X\| \leq \|X - Y\|$$

let $X = X - Y$,

$$\|X\| - \|Y\| \leq \|X - Y\|$$

$$\therefore \big| \|X\| - \|Y\| \big| \leq \|X + (-Y)\| \leq \|X\| + \|Y\|$$

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some examples of norm:

$$\|X\|_1 = |x_1| + \cdots + |x_n|$$

$$\|X\|_2 = (|x_1|^2 + \cdots + |x_n|^2)^{\frac{1}{2}}$$

$$\|X\|_\infty = \sup\{|x_1|, \cdots, |x_n|\}$$

$$\|X\|_1^2 = |x_1|^2 + |x_2|^2 + \cdots + 2|x_1x_2| + 2|x_1x_3| + \cdots$$

$$\sqrt{\|X\|_1^2} \geq \sqrt{(|x_1|^2 + \cdots + |x_n|^2)} = \|X\|_2$$

$$\|X\|_1^2 \leq |x_1|^2 + |x_2|^2 + \cdots + |x_1|^2 + |x_2|^2 + \cdots$$

$$= n(|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2) \qquad \because (|x_1| - |x_2|)^2 \geq 0, \quad 2x_1x_2 \leq |x_1|^2 + |x_2|^2$$

$$\therefore \|X\|_1 \leq \sqrt{n}\|X\|_2$$

$$\|X\|_\infty \leq \|X\|_1$$

$$\|X\|_\infty \leq \|X\|_2$$

$$\|X\|_1 \leq n\cancel{\|X\|_i}^{|x_i|} \leq n\|X\|_\infty \qquad \text{if } \|X\|_i = \sup\{|x_i|, \cdots\}$$

similarly, $\|X\|_2 \leq \sqrt{n}\|X\|_\infty$

$$\therefore \quad \|X\|_2 \leq \|X\|_1 \leq \sqrt{n}\|X\|_2$$

$$\|X\|_\infty \leq \|X\|_1 \leq n\|X\|_\infty$$

$$\|X\|_\infty \leq \|X\|_2 \leq \sqrt{n}\|X\|_\infty$$

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let $|X| = \|X\|_1$

<u>metric space</u>:

$$d : (X, Y) \in V \times V \to d(X, Y) \in \mathbb{R}$$

$$d(X, Y) \geq 0$$

$$d(X, Y) = 0 \text{ iff } X = Y$$

$$d(X, Y) = d(Y, X)$$

$$d(X, Z) \leq d(X, Y) + d(Y, Z)$$

$$d(X, Y) = |X - Y| = |x_1 - y_1| + \cdots + |x_n - y_n|$$

now define convergent sequence X_k,

$$d(X_k, X) \to 0 \quad \text{as} \quad k \to \infty$$

$$\Downarrow$$

$$\lim_{k \to \infty} X_k = X$$

note that if $X_k \to X$, $Y_k \to Y$, $\Rightarrow X_k + Y_k \to X + Y$

$$\alpha X_k \to \alpha X$$

<u>Cauchy sequence</u>:

$$d(X_i, X_j) \to 0 \quad \text{as} \quad i, j \to \infty$$

$\forall \varepsilon$, $\exists M$ such that $i, j > M$, $d(X_i, X_j) < \varepsilon$

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A metric space is complete if $\forall$ Cauchy sequence, $\{X_k\}$, $\exists\, X \in V$ such that $X_k \to X$.

Banach space:

a normed metric space V with metric d.

consider field $\mathbb{R}$ or $\mathbb{C}$, $\mathbb{R}^n$ or $\mathbb{C}^n$ is a Banach space.

$$\because |X_i - X_j| = |X_{i1} - X_{j1}| + \cdots + |X_{in} - X_{jn}|$$

$$\downarrow, \quad \downarrow,$$

$$x_1 \quad X_n$$

$\vdots$

norm of $M_{mn}(\mathbb{F})$

$$|A| = \sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|$$

$$|AB| \le |A||B|$$

---

let V be Banach space, if $\sum X_i$ converges absolutely, it converges.

proof: if $\sum X_k$ converges absolutely, then

$$\sigma_m = \sum_{i=0}^{m-1} \|X_i\| \to \sigma$$

$\exists\, M$, such that $m > M$ $|\sigma_m - \sigma| < \delta$

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let $l > k$,

$$|\sigma_l - \sigma| = |\sigma_k - \sigma + \sum_{i=k}^{l-1} \|X_i\|| < \delta$$

$$|\sigma_k - \sigma| + \sum_{i=k}^{l-1} \|X_i\| \leq |\sigma_k - \sigma| + \sum_{i=k}^{l-1} \|X_i\|| < \delta$$

$$\therefore \sum_{i=k}^{l-1} \|X_i\| < 2\delta = \varepsilon$$

$$\therefore \|\sum_{i=0}^{l-1} X_i - \sum_{i=0}^{k-1} X_k\| \leq \sum_{i=k}^{l-1} \|X_i\| < \varepsilon \quad //$$

$\therefore$ V is Banach space

note that $\sum_k X_{p(k)} \to X$ if $\sum X_k$ converges absolutely

normed vector spaces of functions:

let $C(I, \mathbb{F})$ set of continuous function over $I = [a,b]$

$a, b \in \mathbb{F}$

$$\|f\|_\infty = \sup\{|f(t)|, t \in I\}$$

$$d(f,g) = \|f - g\|_\infty$$

$f_k \to f$: for any $\varepsilon > 0$, $\exists N$ such that $k > N$,

$$\|f_k - f\|_\infty < \varepsilon$$

$\Downarrow$

uniform convergence

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Every ~~th~~ definitions from $C(I, \mathbb{F})$ carried over to $\mathbb{F}(I, M_{nm}(\mathbb{F}))$, let $A = (a_{ij})$

$A(t) \to A(c)$, $t \to c$ continuous, iff a_{ij} is continuous at c.

$$|A(t)| = \sum_{ij} |a_{ij}|$$

$$\|A\|_\infty = \sup\{|A|, t \in I\}$$

$$|a_{ij}| \leq |A(t)| \leq \sum_{ij} \|a_{ij}\|_\infty = \|A\|_\infty$$

$$d(A, B) = \|A - B\|_\infty$$

Analytic functions:

a function is analytic in J, $J = \{t \,|\, |t-\tau| < \rho\}$, $\rho > 0$

if $f(t) = \sum_{k=0}^{\infty} a_k (t-\tau)^k$, $|t-\tau| < \rho$, $a_k \in \mathbb{C}$

if $\sum_k c_k$ converges, and $|a_k| < c_k$, then $\sum a_k$ converges absolutely.

if $\left|\frac{a_{k+1}}{a_k}\right| \to r$, $k \to \infty$, $r < 1$, $\sum a_k$ converges absolutely

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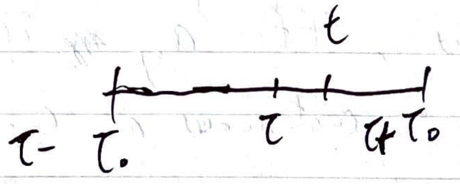
let $f(t) = \sum_{k=0}^{\infty} a_k (t-\tau)^k$, if it converges at $\tau_0 \neq \tau$,

then it converges absolutely and uniformly on $|t-\tau| < |\tau_0 - \tau|$



proof: $\sum_k a_k (t-\tau)^k = \sum a_k (\tau_0 - \tau)^k \frac{(t-\tau)^k}{(\tau_0-\tau)^k}$

$\left| a_k (\tau_0 - \tau)^k \frac{(t-\tau)^k}{(\tau_0 - \tau)^k} \right| \leq M r, \quad r < 1$

$\therefore \sum_k a_k (t-\tau)^k$ converges.

① let $f_n(t)$ continuous functions on $[a,b]$, if $f_n \to f$ uniformly, f is continuous, and

$$\lim_{n\to\infty} \int_a^b f_n(t)dt = \int_a^b \lim_{n\to\infty} f_n(t)dt = \int_a^b f(t)dt$$

② let $f_n(t)$ continuously differentiable functions on $[a,b]$, if $f_n \to f$ pointwise, $f_n' \to h$ uniformly, then f is differentiable and $f' = h$

① $\Rightarrow$ $f(t) = \sum_{k=0}^{\infty} a_k (t-\tau)^k$

$\int f(t)dt = \sum_{k=0}^{\infty} \int a_k (t-\tau)^k dt$

② $\Rightarrow$ $f(t) = \sum_{k=0} a_k (t-\tau)^k$

$f'(t) = \sum_{k=0}^{\infty} (k+1) a_{k+1} (t-\tau)^k$

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$$X' = A(t)X + B(t)$$

assume A and B are analytic at τ, thus for some $\rho > 0$,

$$A(t) = \sum A_k (t-\tau)^k, \quad |t-\tau| < \rho, \quad A_k \in M_n(\mathbb{C})$$
$$B(t) = \sum B_k (t-\tau)^k, \quad |t-\tau| < \rho, \quad B_k \in \mathbb{C}^n$$

without assumption, τ can be set zero by linear transformation.

let $X = \sum C_k t^k, \quad |t| < \rho, \quad C_k \in \mathbb{C}^n$

$$\therefore X' = \sum_{k=1} k C_k t^{k-1} = \sum (k+1) C_{k+1} t^k$$

$$A(t) X = \sum \left(\sum_{j=0}^{k} C_{k-j} A_j \right) t^k$$

$$\therefore X' = A(t)X + B(t) \equiv \sum (k+1) C_{k+1} t^k = \sum \left(\sum_{j=0}^{k} C_{k-j} A_j + B_k \right) t^k$$

$$(k+1) C_{k+1} = \sum_{j=0}^{k} C_{k-j} A_j + B_k,$$

$$C_0 = \xi, \quad \text{initial condition}$$

Since A and B are convergent when $|t| < \rho$, let $0 < r < \rho$,

$$|A_k| r^k < M, \quad |B_k| r^k < M, \quad M > 0$$

$$(k+1)|C_{k+1}| \le M \left(\sum_{j=0}^{k} C_{k-j} r^{-j} + r^{-k} \right)$$

let $d_k = |C_k|$,

$$(k+1)\, d_{k+1} = M r^{-k}\left(\sum_{j=0}^{k} d_{k-j}\, r^{k-j} + 1\right) = M r^{-k}\left(\sum_{j=0}^{k} d_j\, r^j + 1\right)$$

$$k\, d_k = M r^{-k+1}\left(\sum_{j=0}^{k-1} d_j\, r^j + 1\right)$$

$$\therefore (k+1)\, d_{k+1} = \frac{k\, d_k}{r} + M r^{-k} \cdot d_k r^k = M d_k + r^{-1} k\, d_k$$

$$\therefore \left|\frac{d_{k+1}}{d_k}\frac{t^{k+1}}{t^k}\right| = \left|\frac{M}{k+1} + \frac{k}{k+1}\frac{1}{r}\right| |t| \to \frac{|t|}{r} \quad \text{as } k \to \infty$$

applying ratio test, $\sum d_k t^k$ converges.

as $d_k = |C_k|$, $X = \sum C_k t^k$ converges absolutely. //

Examples:

① $X' = AX$, A is a constant matrix.

$$\therefore C_{k+1}(k+1) = \sum_{j=0}^{k} C_{k-j} A_j = C_k A_0 = C_k A$$

$$C_k = \frac{A^k}{k!} C_0 = \frac{A^k}{k!}\xi$$

$$\therefore X(t) = \sum \frac{A^k t^k}{k!}\xi = e^{At}\xi$$

② $X' = (A_0 + A_1 t) X$

$$(k+1) C_{k+1} = C_k A_0 + C_{k-1} A_1, \quad C_0 = I_n, \quad C_1 = A_0$$

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Equations of order n

$$x^{(n)} + a_{n-1}(t)x^{(n-1)} + \cdots + a_0(t)x = b(t)$$

if $a_j(t)$ and $b(t)$ have power series expansion around τ,

$$a_j(t) = \sum a_{jk}(t-\tau)^k$$

$$b(t) = \sum b_k(t-\tau)^k,$$

convergent at $|t-\tau| < \rho$, $\rho > 0$, then given any $\xi \in \mathbb{C}^n$,
$\exists$ a solution $x(t) = \sum c_k(t-\tau)^k$ such that $\tilde{x}(\tau) = \xi$.

proof:

let $y_0 = x$ then $y_{n-1}' + a_{n-1}(t)y_{n-1} + \cdots + a_0(t)y_0 = b(t)$

$y_1 = y_0' = x'$

$y_2 = y_1' = x''$

$\vdots$

$y_{n-1} = y_{n-2}' = x^{(n-1)}$

$$\Downarrow$$

$$Y' = A(t)Y + B(t)$$

$$\begin{pmatrix} y_0 \\ \vdots \\ \vdots \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 & & & \\ 0 & 0 & 1 & & \\ 0 & 0 & 0 & 1 & \\ & & & \ddots & \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{pmatrix} \begin{pmatrix} y_0 \\ \vdots \\ \vdots \\ y_{n-1} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ b(t) \end{pmatrix}$$

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by the previous theorem, since A and B have convergent series, $\exists\ Y = \sum C_k (t-\tau)^k$, such that $Y(\tau) = \xi = \tilde{x}(\tau)$ $\square$

Examples:

Airy equation : $x'' - tx = 0$

let $x = \sum c_k t^k$,

$$x''(t) = \sum_{k=2} k(k-1) C_k t^{k-2} = \sum (k+1)(k+2) C_{k+2} t^k$$

$$t x(t) = \sum_{k=1} C_{k-1} t^k$$

$$\therefore x'' - tx = \sum_{k=1} \left[(k+1)(k+2) C_{k+2} - C_{k-1} \right] t^k + 2C_2 = 0$$

$\Rightarrow C_2 = 0$, $(k+1)(k+2) C_{k+2} = C_{k-1}$

$C_3 = \frac{C_0}{2 \cdot 3}$, $C_4 = \frac{C_1}{3 \cdot 4}$, $C_5 = 0$, $C_6 = \frac{C_3}{5 \cdot 6} = \frac{C_0}{2 \cdot 3 \cdot 5 \cdot 6}$

$C_7 = \frac{C_4}{6 \cdot 7} = \frac{C_1}{3 \cdot 4 \cdot 6 \cdot 7}$, $C_8 = 0$, $C_9 = \frac{C_6}{8 \cdot 9} = \frac{C_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} \cdots$

$\therefore C_{3m} = \frac{C_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3m-1) \cdot 3m}$, $C_{3m+1} = \frac{C_1}{3 \cdot 4 \cdot 6 \cdot 7 \cdots (3m) \cdot (3m+1)}$

$C_{3m+2} = 0$

$\therefore x = C_0 u + C_1 v$, $u = 1 + \sum_{m=1}^{\infty} \frac{t^{3m}}{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3m)}$

$v = t + \sum_{m=1}^{\infty} \frac{t^{3m+1}}{3 \cdot 4 \cdot 6 \cdot 7 \cdots (3m+1)}$

Legendre equation

$$Lx = (1-t^2)x'' - 2tx' + \alpha(\alpha+1)x = 0$$

$$x'' - \frac{2t}{1-t^2}x' + \frac{\alpha(\alpha+1)}{1-t^2}x = 0$$

$$a_0(t) = \frac{\alpha(\alpha+1)}{1-t^2} = \alpha(\alpha+1)\sum t^{2k}$$ convergent for $|t|<1$

$$a_1(t) = \frac{-2t}{1-t^2} = -2t\sum t^{2k} = \sum(-2)t^{2k+1}$$

let $x = \sum c_k t^k$, $x' = \sum (k+1)c_{k+1}t^k$, $x'' = \sum (k+1)(k+2)c_{k+2}t^k$

$$-t^2x'' = -\sum (k+1)(k+2)c_{k+2}t^{k+2} = -\sum k(k-1)c_k t^k$$

$$-2tx' = -2\sum (k+1)c_{k+1}t^{k+1} = -2\sum k c_k t^k$$

thus, $Lx = (1-t^2)x'' - 2tx' + \alpha(\alpha+1)x$

$$= \sum\left[(k+1)(k+2)c_{k+2} - k(k-1)c_k - 2kc_k + \alpha(\alpha+1)c_k\right]t^k$$

$$= \sum\left[(k+1)(k+2)c_{k+2} + (\alpha+k+1)(\alpha-k)c_k\right]t^k$$

$\therefore$ $c_2 = -\frac{\alpha(\alpha+1)}{1\cdot 2}c_0$

$$c_4 = -\frac{(\alpha+3)(\alpha-2)}{3\cdot 4}c_2 = \frac{\alpha(\alpha-2)\cdot(\alpha+1)(\alpha+3)}{1\cdot 2\cdot 3\cdot 4}c_0$$

$$c_6 = -\frac{(\alpha+5)(\alpha-4)}{5\cdot 6}c_4 = -\frac{\alpha(\alpha-2)(\alpha-4)\cdot(\alpha+1)(\alpha+3)(\alpha+5)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}c_0$$

$$c_3 = -\frac{(\alpha+2)(\alpha-1)}{1\cdot 2\cdot 3}c_1 \quad c_5 = -\frac{(\alpha+4)(\alpha-3)}{4\cdot 5}c_3 = \frac{(\alpha+2)(\alpha+4)\cdot(\alpha-1)(\alpha-3)}{1\cdot 2\cdot 3\cdot 4\cdot 5}c_1$$

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$$\therefore C_{2m} = (-1)^m \frac{(\alpha+2m-1)\cdots(\alpha+1)\alpha(\alpha-2)\cdots(\alpha-2m+2)}{(2m)!} C_0$$

$$C_{2m+1} = (-1)^m \frac{(\alpha+2m)\cdots(\alpha+2)(\alpha-1)(\alpha-3)\cdots(\alpha-2m+1)}{(2m+1)!} C_1$$

$$\therefore x(t) = C_0 u + C_1 v$$

$$u(t) = 1 + \sum_{m=1} d_{2m} t^{2m} \qquad v(t) = t + \sum_{m=1} d_{2m+1} t^{2m+1}$$

$$d_{2m} = \frac{C_{2m}}{C_0}, \quad d_{2m+1} = \frac{C_{2m+1}}{C_1}$$

Legendre polynomials

$$p(t) = D^n (t^2-1)^n, \quad D = \frac{d}{dt}$$

a polynomial satisfying $P_n(1) = 1$ of degree n, and $LP_n = 0$, is called the Legendre polynomial.

$p(t)$ is a Legendre polynomial.

proof :

let $v(t) = (t^2-1)^n$

$v' = n(t^2-1)^{n-1} \cdot 2t \implies (t^2-1)v' = 2tnv$

differentiating $(n+1)$ times,

$$(t^2-1)v^{(n+2)} + (n+1)\cdot 2t \cdot v^{(n+1)} + \frac{n(n+1)}{2!}\cdot 2 \cdot v^{(n)} = 2ntv^{(n+1)} + 2n(n+1)v^{(n)}$$

$$\therefore (t^2-1)v^{(n+2)} + 2tv^{(n+1)} - n(n+1)v^{(n)} = 0$$

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$\because p(t) = D^n v = v^{(n)}$

$\therefore (1-t^2)p'' - 2tnp' + n(n+1)p = 0$ $\square$

$p(t) = D^n[(t-1)(t+1)]^n$

$= (t+1)^n D^n(t-1)^n + (t-1)p(t)$

$= n!(t+1)^n + (t-1)p(t)$

$\therefore p(1) = 2^n n!$

$\therefore \boxed{P_n(t) = \frac{1}{2^n n!} D^n (t^2-1)^n}$ ← Legendre polynomial

$P_n(t)$ is the only polynomial solution to the Legendre equation, because:

if $\exists\, q$ of order n which is another solution,

$q = C_0 u + C_1 v \Rightarrow \underbrace{q - C_0 u}_{\text{polynomial}} = \underbrace{C_1 v}_{\text{series}}$ when $n = 2m$

$\therefore P_n = C_0 u$, $P_n(1) = 1 = C_0 u(1) \Rightarrow$ no other non-trivial polynomial solution $q(1) = 1$

uniqueness!! $p_n = P_n \iff p_n - P_n = q_n \equiv 0$

if $\exists\, p_n$ such that $p_n(1) = 1$, $L p_n = 0$

recurrence

$$P_n' - P_{n-2}' = (2n-1)P_{n-1}$$

Proof:

$$P_n' - P_{n-2}' = \frac{1}{2^n n!} D^{n+1}(t^2-1)^n - \frac{1}{2^{n-2}(n-2)!} D^{n-1}(t^2-1)^{n-2}$$

$$= \frac{D^{n-1}}{2^{n-2}(n-2)!}\left[D^2(t^2-1)^n \cdot \frac{1}{4n(n-1)} - (t^2-1)^{n-2}\right]$$

$$= \frac{D^{n-1}}{2^{n-2}(n-2)!}\left[\frac{n(n-1)(t^2-1)^{n-2}\cdot 4t^2 + 2n(t^2-1)^{n-1}}{4n(n-1)} - (t^2-1)^{n-2}\right]$$

$$= \frac{D^{n-1}}{2^{n-2}(n-2)!}\left[(t^2-1)^{n-2}\left[\frac{2n(t^2-1)}{4n(n-1)} + t^2-1\right]\right]$$

$$= \frac{2n-1}{2^{n-1}(n-1)!} D^{n-1}(t^2-1)^{n-1} = (2n-1) P_{n-1} \quad (1)$$

$$\therefore P_{2m}' = (4m-1)P_{2m-1} + (4m-5)P_{2m-3} + \dots + 7P_3 + 3P_1$$

$$P_{2m+1}' = (4m+1)P_{2m} + (4m-3)P_{2m-2} + \dots + 5P_2 + P_0$$

let $n = 2m$,

$$\therefore P_n' = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (2n-4k-1) P_{n-2k-1} \qquad P_0' = 0, \quad P_1' = 1$$

Legendre functions of the second kind

if $\alpha = 0$, $u(t) = 1$

$$v(t) = t + \sum d_{2m+1} t^{2m+1}$$

$$= t + \sum (-1)^m \frac{2\cdot 4 \cdots 2m \cdot (-1)(-3)\cdots(-2m+1)}{(2m+1)!} t^{2m+1}$$

$$= t + \frac{t^3}{3} + \frac{t^5}{5} + \dots = \sum \frac{t^{2m+1}}{2m+1}$$

recall that $\log(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \dots, \quad |t| < 1$

$\log(1-t) = -t - \frac{t^2}{2} - \frac{t^3}{3} - \dots, \quad |t| < 1$

$$\therefore \frac{1}{2}\log\left(\frac{1+t}{1-t}\right) = t + \frac{t^3}{3} + \frac{t^5}{5} + \dots = v(t)$$

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denote $Q_0(t) = \frac{1}{2}\log\left(\frac{1+t}{1-t}\right)$, $|t|<1$

when $\alpha = 1$, $v(t) = P_1$

$$u(t) = 1 + \sum \frac{\overbrace{2\cdot 4 \cdots (2m)\cdot \overbrace{[(-1)\cdot(-3)\cdots(-2m+3)]}}{(2m)!} t^{2m}(-1)^m$$

$$= 1 - t^2 - t^4/3 - t^6/5 - \cdots$$

$$= 1 - t\left(t + \frac{t^3}{3} + \frac{t^5}{5} + \cdots\right) = 1 - tQ_0(t) = 1 - P_1 Q_0$$

denote $Q_1(t) = -u(t) = P_1 Q_0 - 1$

this suggests that $Q_n(t) \overset{?}{=} P_n Q_0 - p$ is a solution?

$\uparrow$ of order $n-1$

$$\begin{aligned} Q_n &= \boxed{P_n Q_0} - p \\ Q_n' &= \boxed{P_n' Q_0} + P_n Q_0' - p' \\ Q_n'' &= \boxed{P_n'' Q_0} + 2P_n' Q_0' + P_n Q_0'' - p'' \end{aligned}$$

$\therefore L_n Q_n = Q_0 L_n P_n + P_n L_0 Q_0 + 2(1-t^2) P_n' Q_0' - L_n p$

$\therefore$ $L_n Q_n = 0$, iff $L_n p = 2P_n'$ because

$L_n P_n = 0$, $L_0 Q_0 = 0$, $Q_0' = \frac{1}{2}\left[\frac{1}{1+t} + \frac{1}{1-t}\right] = \frac{1}{1-t^2}$

before showing the form of p, note that

$$\begin{aligned} L_n P_j &= L_n P_j - L_j P_j \quad \because L_j P_j = 0 \\ &= [n(n+1) - j(j+1)] P_j \\ &= (n+j+1)(n-j) P_j \end{aligned}$$

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and because $P_n' = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (2n-4k-1) P_{n-2k-1}$

$\therefore$

$$= \sum (2n-4k-1) \frac{1}{(n+n-2k\cancel{-1})(2k+1)} \cancel{P} L_n P_{n-2k-1}$$

$$\therefore 2P_n' = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{2n-4k-1}{(n+k)(2k+1)} L_n P_{n-2k-1}$$

$$= L_n p$$

$$\therefore p = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{2n-4k-1}{(n+k)(2k+1)} P_{n-2k-1}$$

if $\exists q$ such that $P_n Q_0 - q$ is another solution,

let $w = p - q \Rightarrow L_n w = L_n p - L_n q = 2P_n' - 2P_n' = 0$

$\Rightarrow w = c_0 u + c_1 v$

$\Rightarrow$ if $n = 2m$, $c_1 \equiv 0$, similarly for $n = 2m+1$

$\Rightarrow c_0 \equiv 0 \because \deg(w) \leq n-1, \deg(u) = n$

$\Rightarrow w = 0, p = q$ uniqueness !!

$Q_0(-t) = q(t) = \frac{1}{2} \log\left(\frac{t+1}{t-1}\right), \quad |t| > 1$

note that $L_0 q = \left[(1-t^2)\frac{d}{dt}\left(\frac{-2}{t^2-1}\right) - 2t \cdot \frac{-2}{t^2-1}\right] \cdot \frac{1}{2}$

$$= \frac{1}{2}\left[\frac{4t}{1-t^2} - \frac{4t}{1-t^2}\right] = 0$$

So, if the original function $Q_0(t)$ is extended to

$Q_0(t) = \frac{1}{2} \log\left|\frac{t+1}{t-1}\right|$, $\quad L_0 Q_0 = 0$ for $|t| \neq 1$

$\because t \to 1 \quad |Q_0(t)| \to \infty$, $\quad c_1 P_n + c_2 Q_n = 0 \Rightarrow c_1, c_2 = 0$

independence !!

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Singular points

$$A_1(t)X' + A_0(t)X = B(t) \quad , \quad X(\tau) = \xi \text{ in interval } I$$

if $A_1(t)$ is invertible at τ, τ is a regular point

if $A_1(t)$ is not invertible at τ, it is a singular point.

$$X' = -A_1^{-1}(t)A_0(t)X + A_1^{-1}(t)B(t)$$

Examples:

$$tx' + x = 0$$

singular point: $\tau = 0$

solution: $x = \frac{c}{t}$, $t \neq 0$

the only solution at $\tau = 0$ is $x = 0$

$$tx' - x = 0$$

singular point : $\tau = 0$

solution $x = ct$

$\because x = ct$ is continuous at $t = 0$,

$\therefore$ infinite solutions satisfying $x(\tau) = 0$

completely different behaviour at $\tau = 0$

consider the problem:

$$(t-\tau)X' = AX \qquad A = \sum A_k(t-\tau)^k$$

$$A_0 \neq 0$$

assume a simpler case where A is a constant matrix, and $\tau = 0$,

$$tX' = AX$$

if $n=1$, $tx' = ax$, $\therefore x = t^a = e^{a\log t}$

$\because x' = e^{a\log t} \cdot \frac{a}{t} = \frac{a}{t}x$

this suggests that $X(t) = e^{A\log t} = t^A$ $t>0$

write $A = QJQ^{-1}$ (Jordan canonical form),

then $X(t) = Qe^{J\log t}Q^{-1} = Qt^JQ^{-1}$

$\because e^{A\log t} = I + A\log t + \frac{A^2}{2!}(\log t)^2 + \dots$

We can also write a new solution basis

$$Y(t) = Qt^J = QX(t)$$

let's investigate the exact form of $Y(t)$.

$Q = \begin{pmatrix} Q_1 & \cdots & Q_n \end{pmatrix}$ n-column vectors.

consider the first J_1 ($r_1 \times r_1$ matrix) with eigenvalue λ_1

$$J_1 = \begin{pmatrix} \lambda_1 & 1 & & 0 \\ & \lambda_1 & 1 & \\ & & \ddots & 1 \\ 0 & & & \lambda_1 \end{pmatrix}, \quad J = \begin{pmatrix} J_1 & & 0 \\ & J_2 & \\ & & \ddots \\ 0 & & & J_m \end{pmatrix}$$

$\because t^J = e^{J\log t} = I + J\log t + \frac{J^2}{2!}(\log t)^2 + \frac{J^3}{3!}(\log t)^3 + \dots$

for Q_1: $Q_1J_1 = \lambda_1 Q_1$

$Q_1J_1^2 = \lambda_1^2 Q_1$

$\therefore Y_1(t) = t^{\lambda_1}Q_1$

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$(Q_1\ Q_2)\begin{pmatrix}\lambda_1 & 1\\ 0 & \lambda_1\end{pmatrix} = (\lambda_1 Q_1,\ Q_1+\lambda_1 Q_2)$

$(\lambda_1 Q_1,\ Q_1+\lambda_1 Q_2)\begin{pmatrix}\lambda_1 & 1\\ 0 & \lambda_1\end{pmatrix} = (\lambda_1^2 Q_1,\ \lambda_1 Q_1 + \lambda_1 (Q_1+\lambda_1 Q_2))$

for Q_2: $(QJ_1)_2 = Q_1 + \lambda Q_2$

$(QJ_1^2)_2 = \underbrace{\lambda Q_1}_{\text{from } QJ_1} + \lambda(Q_1 + \lambda Q_2) = \lambda_1^2 Q_2 + 2\lambda_1 Q_1$

$(QJ_1^3)_2 = \lambda_1^2 Q_1 + \lambda_1(\lambda_1^2 Q_2 + 2\lambda_1 Q_1) = \lambda_1^3 Q_2 + 3\lambda_1^2 Q_1$

$\therefore Y_2(t) = t^{\lambda_1}(Q_1 \log t + Q_2)$

for Q_3: $(QJ_1)_3 = Q_2 + \lambda_1 Q_3$

$(QJ_1^2)_3 = \underbrace{\boxed{\lambda_1^2 Q_2 + Q_1}}_{\text{from } (QJ_1)_2} + \lambda_1(Q_2 + \lambda_1 Q_3)$

$= \lambda_1^2 Q_3 + Q_1 + 2\lambda_1 Q_2$

$(QJ_1^3)_3 = \lambda_1^2 Q_2 + 2\lambda_1 Q_1 + \lambda_1(\lambda_1^2 Q_3 + Q_1 + 2\lambda_1 Q_2)$

$= \lambda_1^3 Q_3 + 3\lambda_1 Q_1 + 3\lambda_1^2 Q_2$

$\therefore Y_3(t) = t^{\lambda_1}\left(Q_1 \frac{(\log t)^2}{2!} + Q_2 \log t + Q_3\right)$

by induction, $Y_{r_1}(t) = t^{\lambda_1}\left(Q_1 \frac{(\log t)^{r_1-1}}{(r_1-1)!} + \cdots + Q_{r_1}\right)$

$\therefore$ a solution basis of $tX' = AX$ can be written as

$$\boxed{X(t) = t^{\lambda_1} P_1(\log t) + \cdots + t^{\lambda_k} P_k(\log t),\quad t > 0}$$

order of $P_i(\log t)$ is at most $m_i - 1$, where m_i is the multiplicity of eigenvalue λ_i.

for $t < 0$,

$X(t) = |t|^{\lambda_1} P_1(\log|t|) + \cdots + |t|^{\lambda_k} P_k(\log|t|),\quad t \neq 0$

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Singular points of the first kind : Special case

$$X' = \left[\frac{R}{t-\tau} + A(t)\right]X \quad , \quad t \neq \tau$$

$$A(t) = \sum A_k (t-\tau)^k \qquad |t-\tau| < \rho \ , \ \rho > 0$$

theorem:

let λ an eigenvalue of R, $\lambda + k$, $k \in \mathbb{Z}$, is not an eigenvalue, $X(t) = |t-\tau|^\lambda P(t)$ is a solution. And,

$$P(t) = \sum P_k (t-\tau)^k \ , \ P_0 \neq 0 \ , \ |t-\tau| < \rho$$

is a convergent series.

proof:

without loss of generality, $\tau = 0$,

$$\therefore \ X' = \left[\frac{R}{t} + A(t)\right]X$$

$$X = |t|^\lambda P(t)$$

$$X' = \lambda |t|^{\lambda-1} P(t) + |t|^\lambda P'(t)$$

$$tX' = \lambda X + |t|^\lambda t P'(t) = RX + tA(t)X$$

$$\therefore \ \lambda P(t) + tP'(t) = RP(t) + tA(t)P(t)$$

$$\lambda P(t) = \lambda P_0 + \sum_{k\geq 1} \lambda P_k t^k$$

$$tP'(t) = t \sum_{k=1} k P_k t^{k-1} = \sum_{k=1} k P_k t^k$$